Representation Theory of Non-Abelian Magnetic Monopoles

Jacopo Uggeri

Supervisor: João Magueijo

Imperial College London, Department of Physics

September 2024

Submitted in partial fulfillment of the requirements for the degree of

Master of Science in

Quantum Fields and Fundamental Forces

Acknowledgements

I would like to thank my supervisor, João Magueijo, for suggesting the topic and then allowing me to explore the suggested topic in my own way, and for his guidance and support throughout the project. I also thank my girlfriend and my family for their support and encouragement throughout the summer.

Abstract

We discuss the representation theory of non-abelian charges in physics, particularly magnetic monopoles, defined analogously to the well known case of electromagnetism. We start by presenting examples of charges and their emergence from physical symmetries. Next, we present concepts of representation theory that allow to generalise the discussion to non-abelian symmetries. We proceed to discuss charge in gauge theories, the distinction between electric and magnetic charges, and their relation through the Dirac quantisation condition. Lastly, using the tools developed in the previous chapters, we discuss how quantisation conditions can be derived for non-abelian charges.

Contents

1 Introduction **2**

1.1 Magnetic monopoles . . . 2

1.2 Generalising charges . . . 3

2 Understanding Charge **5**

2.1 What is Charge? . . . 5

2.2 Charge as a representation . . . 7

2.2.1 Abelian symmetries . . . 8

2.2.2 A non-commutative example: SU(2) . . . 9

2.3 Elements of Representation Theory . . . 11

2.4 Colour and SU(3) . . . 19

2.4.1 Simply connected groups . . . 22

3 Charge in Gauge Theories **24**

3.1 Electromagnetic charges . . . 24

3.1.1 Magnetic monopoles . . . 25

3.1.2 Dirac quantisation . . . 26

3.2 Non-abelian gauge theories . . . 28

3.2.1 Coupling charge to the gauge field . . . 29

3.2.2 Gauge invariant charge . . . 29

3.2.3 GNO quantisation . . . 31

4 Final remarks **34**

4.1 Summary . . . 34

4.2 Last words on gravitation . . . 35

Chapter 1

Introduction

1.1 Magnetic monopoles

Magnetic monopoles have been a topic of interest in theoretical physics for many years, and are some of the most elusive particles physicists have ever tried to detect. The idea of a magnetic monopole is not new, and has been around since the early days of electromagnetism [1]. This is not surprising, as a simple inspection of Maxwell's equations shows that addition of magnetic charge would make them symmetric under the exchange of electric and magnetic fields. Dirac was also concerned with this, and was convinced that monopoles must exist. In 1931, he argued that in quantum mechanics the existence of a single magnetic monopole would imply the quantisation of electric charge, thus providing a theoretical explanation for this known experimental fact [2]. However, despite the theoretical arguments in favour of their existence, monopoles have yet to be detected. This is not for lack of trying, as many experiments have been conducted to detect them. In the eighties, the search for monopoles was a hot topic, since grand unified theories predicted their existence. Polyakov and ′t Hooft showed that magnetic monopoles follows from general principles in theories that unify the fundamental forces [3, 4]. In the late seventies to early eighties, two candidates for monopoles were detected [5, 6]. However, these experiments were never replicated and are now considered to be spurious [7, 8]. In a 1981 letter to Abdus Salam, Dirac wrote that he was disillusioned with the search for monopoles, as no significant experimental progress had occurred since his 1931 paper [9, 10]. Currently, charge quantisation is explained by anomaly cancellation in the standard model [11], or, in the case of grand unified theories, by the compactness of the gauge group [12]. Despite this, and even discounting grand unified theories, there are valid reasons to believe that monopoles exist. First of all, monopoles would most likely be very massive particles, with an estimated mass of 10^{16}GeV. They would be extremely hard to detect, both in the laboratory and even from cosmic rays [13, 8]. The only events that could most likely generate such massive particles would have happened in the early universe, making their detection a cosmological issue [14, 15]. In fact, the study of monopoles even led to advancements in cosmology, such as the inflationary model [16, 17]. However, magentic charge is not a concept that has to be restricted to electromagnetism. Other theories, such as the one of the strong interaction, are also described by a gauge principle, and have a similar structure to electromagnetism, the difference being that their gauge group is non-abelian. The additional degrees of freedom in these theories behave roughly like multiple electromagnetic fields, but the non-commutativity of the group makes the situation more complex, as the fields can interact with each other. It is worth studying how magnetic monopoles can be understood in these theories, and what their properties are. They might in fact tell us much about the structure of the theory itself.

1.2 Generalising charges

There are several questions that arise in the attempt to generalise magnetic charge to non-abelian gauge theories and gravity. One of which is perhaps the most obvious, if we are to even begin, is to ask what qualities of magnetic charge must be generalised, and which ones are instead unique to electromagnetism. The presence of multiple generators in non-abelian gauge theories offers the opportunity for more structure, but their non-commutativity brings additional restrictions. For example, non-commuting generators cannot be simultaneously diagonalised, which means that not all charge operators are compatible. However, before considering magnetic charge, it is important to understand what is meant by charge in general, and how the more familiar electric charge is understood, as even this is not immediately obvious. As an example, consider the case of the strong interaction, which can be understood as a gauge theory of the group SU(3). In this case, there is not a single charge, but a triplet of "colour" charges. Anti-particles carry anti-colour charge, and the force is mediated by vector bosons - gluons - which carry both colour and anti-colour charge [18]. There are, as we would expect, 8 gluons, as SU(3) has 8 generators. One would expect there to be 8 charges, one for each generator. Why then do we only see 3 charges and 3 anti-charges? And why do gluons carry both a charge and an anti-charge? The discussion in Chapter 2 will shed some light on these questions, clarifying how charge can be understood for non-abelian groups through the language of representation theory. We will then move on to gauge theories in Chapter 3, where charge plays a central role, and "electric" and "magnetic" charges can be defined. Through the language of representation theory, Dirac's quantisation condition can be beautifully extended to the non-abelian case, and offer some insights on the structure of gauge theories.

Chapter 2

Understanding Charge

2.1 What is Charge?

Physicists might talk about charge in different contexts, but there is one property that is always present: it is a conserved quantity. Undergraduate students first encounter this when studying electromagnetism, where the conservation of charge is expressed by the continuity equation:

$$\partial_\mu j^\mu = 0 \tag{2.1}$$

Later the student might learn that this is a consequence of Noether's theorem, which states that for every continuous symmetry of the action there is a conserved quantity.

Noether's theorem (field theory)

Given a continuous symmetry of the action $S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)$, the conserved current is given by:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \mathcal{L} \delta x^\mu \tag{2.2}$$

where $\delta\phi$ is the transformation of the field and δx^μ is the transformation of the coordinates. The conservation of the current is expressed by the continuity equation (2.1).

In the case of electromagnetism, this is a U(1) gauge symmetry. This is the first of many results in physics which link symmetry and conservation laws, hinting at a deep connection between the two. The same can be seen in a Hamiltonian formulation of classical mechanics, where the conserved quantities are the generators of the symmetries of the system. If a quantity is conserved in time, its Poisson bracket with the Hamiltonian is zero:

$$\frac{dA}{dt} = \{A, H\} = 0 \tag{2.3}$$

This quantity also generates a symmetry transformation of the system which leaves the Hamiltonian invariant:

$$\delta q = \epsilon\{q, A\} \tag{2.4}$$

$$\delta p = \epsilon\{p, A\} \tag{2.5}$$

And through the correspondence principle [19], we can see that the same is true in quantum mechanics, where we get conserved quantities from operators that commute with the Hamiltonian.

Correspondence principle

The correspondence principle states that the classical limit of a quantum system is given by the expectation values of the operators in the quantum system. In particular, the commutator of operators in the quantum system corresponds to the Poisson bracket of the corresponding classical quantities:

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \rightarrow \{A, B\} \tag{2.6}$$

This is most obvious in the Heisenberg picture, where the time evolution of an operator is given by the commutator with the Hamiltonian.

$$\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}[\hat{H}, \hat{A}] \tag{2.7}$$

As charges are compatible with time evolution, the eigenvalues of these operators are normally used to define "quantum numbers" that label the states of a system. Thus, both from classical and quantum mechanics, we know that:

- charge is a conserved quantity
- charge generates a symmetry of the system
- charge provides a label for the states of the system

There is a great mathematical tool that allows to understand these properties in a more general context, and that is representation theory. Quantum mechanically a symmetry is represented by a unitary operator $\hat{U}(g) = \exp(i\hat{Q})$ that commutes with the Hamiltonian: $[\hat{U}(g), \hat{H}] = 0$. The infinitesimal generator $\hat{Q}$ of the symmetry is Hermitian and has constant eigenvalues, thus for each symmetry we have a set of Hermitian operators that commute with the Hamiltonian, and so are conserved quantities. As seen in the next section, $\hat{U}(g)$ is an element of a unitary representation of the symmetry group, while $\hat{Q}$ are from a representation of the Lie algebra of the group.

2.2 Charge as a representation

The fundamental relationship between charge and symmetry stands in representation theory [20]. Representation theory concerns itself with the study of group actions on vector spaces, and provides a powerful tool to understand the symmetries of a system. In the context of quantum mechanics, we care about unitary representations [19], and the vector space in question is the Hilbert space $\mathcal{H}$ of the system.

The power of representation theory is that it allows to classify how different states transform under the action of the symmetry group, thus allowing a priori statements on the objects allowed in a theory based on general considerations. We will see that charge appears naturally as such a label. For elements close to the identity $g \approx e$, a representation can be expanded as:

$$\pi(g) = \mathbb{1} + iQ \approx e^{iQ} \tag{2.8}$$

Where the matrix $Q \approx 0$. For a unitary representation $\pi(g)^\dagger \pi(g) = \mathbb{1}$, so $Q = Q^\dagger$ is Hermitian. Thus, the unitary representation of the action of G on a quantum mechanical system naturally provides a set of Hermitian operators Q on $\mathcal{H}$. Given a continuous symmetry in quantum mechanics, we automatically get a set of physical observables! What is more remarkable is that many observables that arise this way are of physical interest [20]. As they carry out infinitesimal group actions on the Hilbert space, these observables live in representations of the Lie algebra of the group, thus, when physicists say a particle is charged under a symmetry, they mean it transforms under a certain representation of the group, labelled by the charge. However, this is not yet the full picture: there are additional complexities that we have not yet addressed. These will be explored in the context of non-commutative symmetries, in Section 2.3.

2.2.1 Abelian symmetries

Let us consider the simplest case of a continuous abelian, or commutative, symmetry. Examples of such a symmetry are the time symmetry in quantum mechanics, or the U(1) phase symmetry in electromagnetism. For a simple abelian symmetry, there is only one generator, the charge operator $\hat{Q}$. The Hilbert space $\mathcal{H}$ here is 1 dimensional and the unitary representations of G are given by:

$$\pi(\theta) = e^{i\theta\hat{Q}} \tag{2.9}$$

Writing the action of $\pi(\theta)$ on a state $|\psi\rangle$ as $\pi(\theta)|\psi\rangle = e^{i\theta\hat{Q}}|\psi\rangle$, we identify the eigenvalues of $\hat{Q}$ as the charges of the system: $\hat{Q}|q\rangle = q|q\rangle$. For a U(1) phase symmetry, we know $e^{i2\pi q}|\psi\rangle$ is physically equivalent to $|\psi\rangle$, which implies $q \in \mathbb{Z}$, thus the allowed values of the charge are quantised [1]. Meanwhile, the time symmetry has no such restriction, and the charge can take any real value. The difference between these two symmetries is in the global features of the group, which is compact in the former and non-compact in the latter. This is reflected in the allowed values of the charge, which are continuous in the former and quantised in the latter.

2.2.2 A non-commutative example: SU(2)

With the group SU(2) things get more interesting. Its Lie algebra is the simplest non-abelian Lie algebra, and has 3 generators. Angular momentum arises as the conserved charge in the presence of an SO(3) spatial rotational symmetry. Quantum mechanically, rotations are described by a unitary operator $\hat{U}(\theta, \phi, \psi)$ which is a representation of an SO(3) element and can be obtained by the exponentiation of infinitesimal rotations, generated by the Hermitian operators $\hat{J}_i$, the angular momentum operators. These form a representations of the Lie algebra $\mathfrak{so}(3)$, and satisfy the commutation relations [21]:

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k \tag{2.10}$$

In principle, as discussed in Section 2.2, these operators correspond to physical observables of the system. However, the relations (2.10) imply that they cannot be simultaneously diagonalised, and thus only one component of angular momentum

[1] If we allow for projective representations, then this is not the case.

can be measured at a time. There is however another Hermitian operator that commutes with all the $\hat{J}_i$, the Casimir operator:

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 \tag{2.11}$$

Since $\hat{J}^2$ commutes with all the $\hat{J}_i$, it commutes with all group elements and is thus invariant under the action of the group. By Schur's lemma [22], it must be a multiple of the identity for any irreducible representation. A Hilbert space can thus be constructed with the following procedure: define the (not Hermitian) operators $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$, which satisfy the commutation relations:

$$[\hat{J}_3, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm \tag{2.12}$$

These form a basis for the complexified Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, we will see later that this complexification is important. They have the property that for a $\hat{J}_3$ eigenstate $\hat{J}_3 \left|m\right\rangle = m \left|m\right\rangle$, the action of $\hat{J}_\pm$ is:

$$\hat{J}_3\hat{J}_\pm \left|m\right\rangle = (m \pm 1)\hat{J}_\pm \left|m\right\rangle \tag{2.13}$$

meaning that the action of $\hat{J}_\pm$ is to raise or lower the eigenvalue of $\hat{J}_3$ by one unit. If we require a finite dimensional representation of $\mathfrak{so}(3)$, there must be a maximum value j of m for which $\hat{J}_+ \left|j\right\rangle = 0$. This provides a label for the representation and can be proven that this label is unique. Similarly, there must be a minimum value $-j'$ for which $\hat{J}_- \left|-j'\right\rangle = 0$. Since the algebra (2.12) is symmetric under the exchange $\hat{J}_+ \leftrightarrow \hat{J}_-$, we have that $j = j'$. Equation (2.12) and the trivial $[\hat{J}_z, \hat{J}_z] = 0$ tell us that the complex vector space spanned by $\hat{J}_\pm, \hat{J}_z$ is split into eigenspaces of $\hat{J}_z$. By Schur's lemma, these are 1-dimensional, and the representation is completely determined by the highest weight state. Thus, we can distinguish two pieces of information:

- the maximum value of the charge j depends on the representation
- the allowed values of the charge are the eigenvalues m of $\hat{J}_z$ which range from $-j$ to j.

2.3 Elements of Representation Theory

To better understand these concepts, it is useful to introduce the language of representation theory, which concerns itself with the study of group actions on vector spaces. Here, we will focus on representations of Lie groups, as mathematical objects used to describe continuous symmetries of a system.

Lie Group

Definition 2.3.1. *A Lie Group* G *is a set of transformations labelled by continuous parameters that forms a group under composition. That is, there is a binary operation* $\mathrm{G} \times \mathrm{G} \to \mathrm{G}$ *that is continuous and associative, an identity element* $e \in \mathrm{G}$ *and an inverse* $g^{-1} \in \mathrm{G}$ *for each* $g \in \mathrm{G}$ *such that* $gg^{-1} = e$.

While groups are abstract mathematical objects, we would like to study how they emerge in real physical systems. Representation theory is the study of how groups can be concretely realised.

Representation of a Lie Group

Definition 2.3.2. *A finite dimensional complex representation* (π, V) *of a Lie group* G *is a map* $\pi : \mathrm{G} \to \mathrm{GL}(n, \mathbb{C})$ *that preservers the group structure (it is a homomorphism).* V *is a* n *dimensional vector space the representation acts upon, also called a* $G-$*module. For an element* $g \in \mathrm{G}$*, its representation acts on* $v \in V$ *as* $\pi(g)v$*. The representation is unitary if* $\pi(g)$ *is unitary for all* g*. A representation is faithful if* π *is injective, that is, if there is an isomorphism between* G *and* $\pi(G)$ *as a group.*

Each G-module defines a representation of the group, which can be explicitly realised by choosing a basis for V An important notion is the one of reducibility, which allows decomposing a representation into simpler ones. If some subspace of the module is invariant under the action of the group, then the representation is reducible.

Irreducible representation

Definition 2.3.3. *A representation* (π, V) *is irreducible if there are no non-trivial invariant subspaces of* V *under the action of* G.

Reducible (unitary) representations can be decomposed into a direct sum of irreducible representations [22], hence by classifying the irreducible representations of a group we can classify all representations. As a concrete example of representations, consider the group SO(3), the group of rotations in 3 dimensions. Representation theory can tell us all the way an object can transform under rotations: some objects do not change, like a sphere, which will be mapped by itself under any rotation, they are said to be in the trivial representation. A cylinder instead will be mapped to itself only under rotations around its axis, thus it transforms under a different representation of the group. There can also be objects are not invariant under any rotation, like a vector, which will be mapped to a different vector under any rotation, this is the first faithful representation of the group. Objects obtained from two vectors, like a dot product or a vector product, will transform under yet different representations of the group. And so on. Infinitesimal transformations as seen at the start of 2.2 can be described by the Lie algebra of the group:

Lie algebra

Definition 2.3.4. *The Lie algebra* $\mathfrak{g}$ *of a Lie group* G *is the tangent space at the identity* $T_e\mathrm{G}$*. It is a vector space with a binary operation called the Lie bracket* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*. The Lie bracket satisfies the Jacobi identity and is antisymmetric.*

The connected component of the identity of a Lie group is uniquely determined by its Lie algebra. This is a consequence of the exponential map, which maps the Lie algebra to the group: $g = e^X$ for $X \in \mathfrak{g}$. By choosing a basis T^i for the Lie algebra, we see the group is generated by these elements $g = e^{i\theta_i T^i}$.

The generators essentially represent directions in the group manifold. Lie algebras also have representations, which are defined analogously to group representations. Furthermore, a G-module is also a $\mathfrak{g}$-module, since the action of the group on the module can be exponentiated to the action of the Lie algebra:

$$\pi(g)v = e^{iX}v \tag{2.14}$$

We have infinitesimally:

$$\pi(g)v = \mathbb{1}v + iXv \tag{2.15}$$

The Lie algebra is also its own module, with an action given by the Lie bracket. This corresponds to the adjoint representation of the group. One thing has to be clear, representation theory does not tell us what the object is, or if it exists at all. It can only tell us all the types of objects that are allowed by the symmetry of a theory. Nevertheless, it is an extremely powerful tool at our disposal to study possible theories and their particle content.

Simple Lie algebras

It is useful to define a special class of Lie algebras

Simple Lie algebra

Definition 2.3.5. *An ideal is a subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *such that* $[H, X] \in \mathfrak{h}$ *for any* $H \in \mathfrak{h}$ *and* $X \in \mathfrak{g}$.

Definition 2.3.6. *A Lie algebra is simple if it has no non-trivial ideals.*

A semi-simple Lie algebra can thus be defined as a direct sum of simple Lie algebras:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n \tag{2.16}$$

Thanks the work of Cartan and Killing, we have a full classification of the complexified semi-simple Lie algebras and their irreducible representations [23]. Their classification has different ingredients, which we will now introduce.

Cartan subalgebra

Definition 2.3.7. *The Cartan subalgebra* $\mathfrak{h}$ *of a Lie algebra* g *is defined as a maximal abelian subalgebra:*

- $[H_i, H_j] = 0$ *for all* $H_i, H_j \in \mathfrak{h}$
- *if* $[H_i, X]$ *= 0 then* $X \in \mathfrak{h}$

The rank r of a Lie algebra is the dimensions of the Cartan subalgebra, equivalently, and more relevant for quantum mechanics, it is the maximum number of mutually commuting generators!

Since G has r commuting generators, it is useful to consider the simultaneous eigenvectors under some representation: These define the invariant subspaces of $\mathfrak{g}$ under the action of $\mathfrak{h}$, which are called weight spaces.

Weight space

Definition 2.3.8. *Given a representation (π, V) of the Lie algebra $\mathfrak{g}$, and a vector $v \in V, v \neq 0$ such that:*

$$\pi(H_i)v = m_i v \tag{2.17}$$

for each generator of the Cartan subalgebra $H_i, i \in \{1 \ldots r\}$. The vector v is called a weight vector, and the space of all v for a given weight $\mu = \{m_i\}$ is its weight space. The multiplicity of a weight is the dimensions of the weight space.

The weight μ can be considered a vector in the dual space of the Cartan subalgebra, $\mathfrak{h}^*$. Its associated weight space is denoted V_μ. We note that since we are considering complexified Lie algebras, and irreducible complex representations of abelian Lie algebras are 1-dimensional, the weight spaces are all 1-dimensional. This is the reason we considered the complexification of the Lie algebra. The Lie algebra itself is a module for the group, and the representation that acts upon it is the adjoint representation, with the action being the lie bracket. This allows to define the adjoint weights $\alpha = \{a_i\}$:

$$[H_i, X] = a_i X \tag{2.18}$$

for $X \in g$. The nonzero adjoint weights are conventionally called roots, while the elements X are called root vectors. The cartan generators H_i are weight vectors with weight 0, so conventionally they are excluded from the root vectors. The roots allow the decomposition of the Lie algebra into a direct sum of root spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \tag{2.19}$$

Cartan's theorem states that the highest weight of an irreducible representation uniquely classifies it, thus we can describe all irreducible representations by their highest weight [22, 23]. This is what allowed us in the case of SU(2) in Section 2.2.2 to classify the representations by the maximum value of the charge. An important lemma [22] allows to generalise raising and lowering operators from Section 2.2.2:

Lemma 2.3.9. *Let* $\alpha = \{a_i\}$ *and its corresponding root vector* X_α*, let* π *be a representation of the Lie algebra* g *and* $\mu = \{m_i\}$ *be a weight for* π *and* $v \neq 0$ *be its root vector*

$$\pi(H_i)\pi(Z_\alpha)v = (m_i + \alpha_i)\pi(Z_\alpha)v \tag{2.20}$$

Thus either $\pi(Z_\alpha)v = 0$ *or* $\pi(Z_\alpha)$ *is a new weight vector with weight* $\mu + \alpha$

This can be easily seen by applying the commutation relations (2.18) to the representations of H_i and X. We can thus deduce three things:

- weights are real, since they are eigenvalues of Hermitian operators
- weights are quantised
- weights label subspaces of the Hilbert space that are invariant under the action of the Cartan subalgebra

Killing form

We can define an inner product on the Cartan subalgebra by:

$$\langle X, Y \rangle = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) \tag{2.21}$$

Where $\mathrm{ad}(X)$ refers to the adjoint representation of the element X. This is called the Killing form. It also induces an inner product on the dual space of the Cartan subalgebra, $\mathfrak{h}^*$:

$$\langle \mu, \nu \rangle = \sum_i \mu_i \nu_i \tag{2.22}$$

Since in the next sections we will be using different kind of bilinear forms, it is useful to clarify their difference and distinguish them.

- $\langle \mu, H \rangle = \mu(H)$ is the action of an element $\mu \in \mathfrak{h}^*$ on an element $H \in \mathfrak{h}$ defined by (2.17)
- $\langle \mu, \nu \rangle$ is the inner product of two elements of $\mathfrak{h}^*$ defined by (2.22)
- $\langle X, Y \rangle$ is the inner product of two elements of $\mathfrak{g}$ defined by (2.21)

Cartan-weyl basis

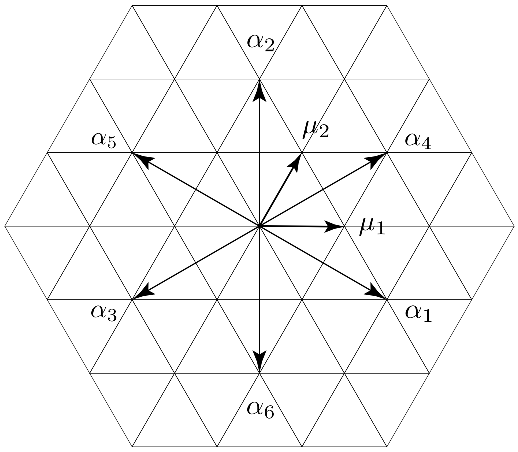

Figure 2.1: The root diagram of $\mathfrak{su}(3)_{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})$.

A Cartan-weyl basis is a convenient choice of basis for a Lie algebra that makes the roots explicit in the structure constants. For a rank r Lie algebra we have r H_i Cartan generators and $d - r$ E_α remaining generators. The commutation relations are:

$$[H_i, H_j] = 0 \tag{2.23}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{2.24}$$

Using the Killing form and the Cartan-weyl basis, we can define the weight and root diagram of a group by plotting the real subspace of $\mathfrak{h}^*$ spanned by the weights and roots. This is shown in 2.1 for the case of $\mathfrak{su}(3)$.

Weight and root lattice

The weights of a Lie algebra span a lattice in $\mathfrak{h}^*$, defined by the inner product (2.21). This is the weight lattice Λ_μ. Similarly, the roots span a lattice Λ_α, which is a sublattice of the weight lattice.

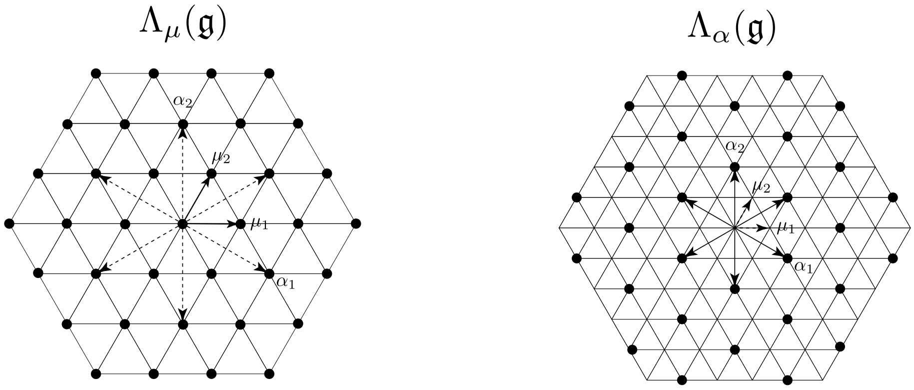

Figure 2.2: The weight and root lattice of $\mathfrak{su}(3)$, generated respectively by the fundamental weights $\mu_{1,2}$ and the simple roots $\alpha_{1,2}$.

Weyl group

The Weyl group is a discrete group that acts on the weight space of a Lie algebra. It is generated by the reflections through the hyperplanes orthogonal to the roots. An element of the Weyl group associated to a root α is denoted s_α and is defined by:

$$s_\alpha(\mu) = \mu - \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \tag{2.25}$$

Coroot lattice

We can also define co-roots by the property:

$$\langle \alpha^\vee, \alpha \rangle = 2 \tag{2.26}$$

Which gives the co-roots as:

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \tag{2.27}$$

Equation (2.26) allows us to recognise the co-roots as elements of the dual space of $\mathfrak{h}^*$, which is the Cartan subalgebra itself. Since the co-roots generate the Weyl group reflections in (2.25) which map the weight space to itself, we must have:

$$s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha \in \mathfrak{h} \tag{2.28}$$

And since the weights differ by integral multiples of the roots, we have the property:

$$\langle \mu, \alpha^\vee \rangle \in \mathbb{Z} \tag{2.29}$$

Co-roots define the co-root lattice $\Lambda_{\alpha^\vee}$, which is the dual lattice of the root lattice.

2.4 Colour and SU(3)

The above concepts help us generalise what we saw in 2.2 to the case of other non-abelian symmetries. Here we will consider the case of SU(3) and give a concrete example of how charges arise from this symmetry. We saw how for SU(2) the set of mutually commuting generators has 1 element J_z, which allows to split the Hilbert space into eigenspaces of J_z labelled by their charge. The same happens for SU(3), but now the largest set of commuting generators has 2 elements, which we call H_1 and H_2. This is the Cartan subalgebra of $\mathfrak{su}(3)$. We choose the basis [22]:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.30}$$

Now each state is labelled by its simultaneous eigenvalues of H_1 and H_2 and the charge becomes an ordered pair of numbers $\mu = (m_1, m_2)$. We can explicitly find eigenstates in the Hilbert space of a particle in the fundamental representation:

$$v = \begin{pmatrix} r \\ g \\ b \end{pmatrix} \tag{2.31}$$

Giving the eigenvalues:

$$\begin{aligned} v &= \begin{pmatrix} r & 0 & 0 \end{pmatrix} \quad \text{with} \quad \mu_r = (1,0) \\ v &= \begin{pmatrix} 0 & g & 0 \end{pmatrix} \quad \text{with} \quad \mu_g = (-1,1) \\ v &= \begin{pmatrix} 0 & 0 & b \end{pmatrix} \quad \text{with} \quad \mu_b = (0,-1) \end{aligned} \tag{2.32}$$

These justify the choice of the rgb labels for the components, as they correspond to eigenstates with a specific "colour" charge. From a representation theory perspective, the Lie algebra representation (SU(3) fundamental) can be split into multiple representations irreducible under the subalgebra h spanned by H_1 and H_2:

$$\mathbf{3} = \mathbf{1} \oplus \mathbf{2} \tag{2.33}$$

The charges also have the expected property that a bound state of rgb is in the trivial representation (has weights $(0,0)$) as seen by taking $\mu_r + \mu_g + \mu_b = 0$ We can also choose the anti-fundamental representation:

$$\bar{v} = \begin{pmatrix} \bar{r} \\ \bar{g} \\ \bar{b} \end{pmatrix} \tag{2.34}$$

On which $H_{1,2}$ acts as $-H_{1,2}^T$. Giving:

$$\begin{aligned} \bar{v} &= \begin{pmatrix} \bar{r} & 0 & 0 \end{pmatrix} \quad \text{with} \quad \mu_{\bar{r}} = (-1,0) \\ \bar{v} &= \begin{pmatrix} 0 & \bar{g} & 0 \end{pmatrix} \quad \text{with} \quad \mu_{\bar{g}} = (1,-1) \\ \bar{v} &= \begin{pmatrix} 0 & 0 & \bar{b} \end{pmatrix} \quad \text{with} \quad \mu_{\bar{b}} = (0,1) \end{aligned} \tag{2.35}$$

One can also note $\mu_r + \mu_{\bar{r}} = \mu_g + \mu_{\bar{g}} = \mu_b + \mu_{\bar{b}} = 0$ meaning a $c\bar{c}$ state is bound. Also note that the colours and anti-colours are not independent, for example $\mu_r + \mu_g = \mu_{\bar{b}}$. From this we can notice that the SU(3) adjoint representation can be built from the fundamental and the anti-fundamental $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ which leads to the conclusion that gluons (particles in the adjoint representation) should have a colour and an anti-colour. In fact we could choose a basis which makes color explicit for the gluon gauge fields A_μ^a by considering the generators $T_{c\bar{c}}^a$ (which have a colour and anti-colour index because of the above) so that $A_\mu^a T_{c\bar{c}}^a = A_{\mu c\bar{c}}$ In the SU(2) case above the gauge field would instead have two isospin charges as (for SU(2)) $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$.

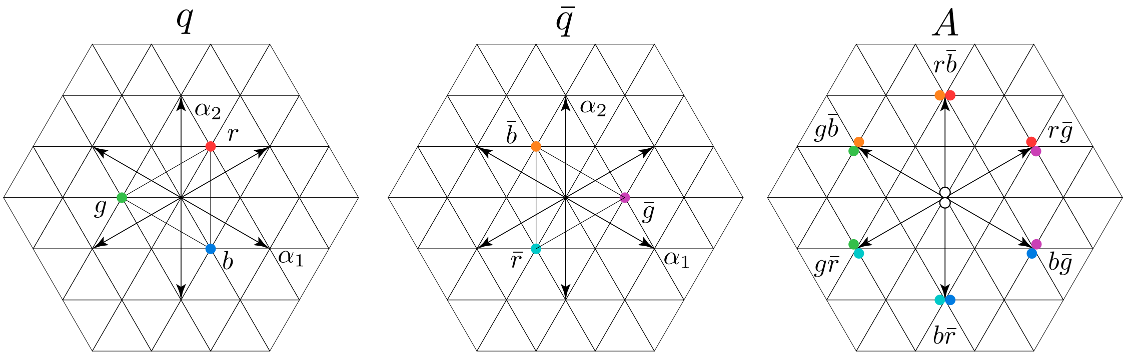

Figure 2.3: The weight diagrams of the fundamental, anti-fundamental and adjoint representations of $\mathfrak{su}(3)$, marked with colour labels to show the correspondence with the rgb charges and their anti-charges. These sum as vectors in $\mathfrak{h}^*$, showing that adjoint states have both a charge and an anti-charge. Although there are three combinations that give a weight of $(0, 0)$, only two of there are linearly independent, as they correspond to linear combinations of the the (degenerate) adjoint weights of the Cartan subalgebra generators.

There is no anti-spin because $\mathbf{2} \equiv \bar{\mathbf{2}}$. This analysis shows that a consistent candidate for charge would be the set of weight eigenvalues. While a field can be "charged" under a group, that is transform in a certain representation, the actual value of the charge is the weight, which is conserved as the generators of $\mathfrak{h}$ commute with the hamiltonian. They correspond to subspaces of the Hilbert space, which are invariant under the action of the Cartan subalgebra. Under the right choice of basis for the Hilbert space the charge eigenstates can be made to correspond with the components of the chosen representation like in the case above. Different representations like the adjoint or other higher dimensional ones can be built from the fundamental, anti-fundamental and/or the adjoint, and their charges can be found by a simple addition of weights. The name "fundamental" weights is telling because they are the simplest charges that can be found under a symmetry. A discussion can be carried out for any Lie group, including product groups like the Lorentz group which has rank 2 Lie algebra $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The group representations are labelled with two spins (j_1, j_2)

and correspond to right and left handed spinors, vectors, self-dual and anti-self-dual 2-forms, etc.

2.4.1 Simply connected groups

A theorem by Cartan and Killing [22, 23] states that complex Lie algebras are semisimple if and only if they are isomorphic to the Lie algebra of a compact and simply connected Lie group. This result implies that a general compact Lie group can be written as:

$$\mathrm{G} = \frac{\mathrm{G}_s \times \mathrm{U}(1)^n}{\mathrm{Z}} \tag{2.36}$$

Where G_s is a simply connected group and Z is a discrete subgroup of its centre.

Centre of a group

Definition 2.4.1. *The centre of a group* G *is the set of elements that commute with all elements of the group:*

$$\mathrm{Z} = \{z \in \mathrm{G} | zg = gz \quad \forall g \in \mathrm{G}\} \tag{2.37}$$

It is a normal subgroup of G*.*

It is important to note the physical implications of this result, particularly the possibility of a quotient by a discrete subgroup. At the level of the Lie algebra the groups G and G/Z are isomorphic: the difference is all in the global properties of the group. However, this can very well have physical implications, as the global properties determine the allowed representations of the group. We encountered this when discussing the groups SO(3) and SU(2), corresponding to angular momentum and spin. The only way to distinguish between the two is that we know experimentally that the half-integer representations of SU(2) are physically realised in the case of spin. In the case of angular momentum instead they do not correspond to any physical object [21], and only representations that are invariant under the centre Z_2

are allowerd. A similar situation can be found in the case of SU(3), and is relevant for Yang-Mills theory. Pure Yang-Mills theory only has gauge bosons, which are in the adjoint representation of the group, these are invariant under the action of the center of the group, thus the gauge group of Yang-Mills could be $SU(3)/Z_3$. However in the standard model there are also quarks in the fundamental and anti-fundamental representation, which are not invariant under the centre of the group, thus the gauge group of the strong interaction in the standard model must be SU(3).

Chapter 3

Charge in Gauge Theories

3.1 Electromagnetic charges

So far charge has been described purely in terms of the representation theory of a given group. Energy, momentum, angular momentum, spin and particle number are all examples of charges under this general definition. However, the charges involved in the fundamental interactions have more to them than just the representation theory. It is only in this context that a distinction between *electric* and *magnetic* charge is possible. Coming from classical electromagnetism, nothing other than experimental evidence suggests that magnetic monopoles shouldn't exist. Maxwell's equations read [24]:

$$\nabla \cdot E = \rho \qquad \nabla \times E + \dot{B} = 0 \tag{3.1}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B - \dot{E} = j \tag{3.2}$$

Or in relativistic notation:

$$\partial_\mu F^{\mu\nu} = -j^\nu \tag{3.3}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{3.4}$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is its dual, and j^ν is the 4-current. As we can see from the first equation in (3.1), a point-like charge q at the origin sources an electric field $E = \frac{q}{r^2}\hat{r}$. Analogously, a hypothetical magnetic monopole g would source a magnetic field $B = \frac{g}{r^2}\hat{r}$. However, this is prohibited by the first equation in (3.2), which states that the divergence of the magnetic field is always zero. We note the suggestive duality in the equations (3.1) and (3.2) under the exchange $E \longleftrightarrow B$ when sources are not present. Quoting Gell-Mann "Everything that is not forbidden is compulsory" [25]. It is extremely tempting to extend this duality to the case with sources, and nothing stops us from a priori introducing a magnetic current k^ν

$$\partial_\mu \star F^{\mu\nu} = k^\nu \tag{3.5}$$

When the theory is formulated from a gauge principle however, this duality between electricity and magnetism is broken. By positing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ Now the homogeneous equations are simple mathematical identities, thus magnetic charge is incompatible with a globally defined gauge potential. Here lies one of the difficulties in the construction of a theory of magnetic monopoles, as in this formulation the distinguishing feature of magnetic charge relies on its most undesirable properties. It is unclear how one would go about defining an action whose equations of motion include equation (3.5).

3.1.1 Magnetic monopoles

Although a description of electromagnetism using a gauge potential A_μ initially seems incompatible with magnetic monopoles, there are in fact ways to allow for non-zero magnetic flux. One such method was proposed by Dirac in 1931, which involved the introduction of a singular gauge potential. By allowing the gauge potential to take different values on either side of a singular line of points, Dirac interpreted this singularity physically as an infinitely thin solenoid extending to infinity. This concept is related to the fact that the Bianchi identities (3.4) can be satisfied locally but not globally, due to non-trivial topologies that cause the field strength to be non-closed. Monopoles can also arise in grand unified theories (GUTs), where they appear as topological solitons, as shown by ′t Hooft and Polyakov in 1974 [3, 4]. They also feature in string theory [26] and Kaluza-Klein theory [27]. Regardless of the specific mechanism by which magnetic charge is introduced, we will see that representation theory allows us to make general statements about the compatibility of electric and magnetic charges.

3.1.2 Dirac quantisation

Dirac was a strong proponent of magnetic monopoles for another reason: the existence of a magnetic monopole would imply the quantisation of all electric charge [2, 28]. Since, as far as we know, charge in our universe is quantised, this would make the presence of magnetic monopoles a likely explanation. Dirac's argument used the Dirac string, showing that the requirement for it to be unobservable, and thus unphysical, is exactly the quantisation condition. Corrigan and Olive coupled an adjoint scalar to the gauge field to generate charged current [29]. Here we present a simpler argument using the path integral formalism, adapted from Heras' [30] and Tong [31], which considers the Aharonov-Bohm [32] phase gained by a moving charge. We consider the path integral phase picked up by an infinitely massive probe particle with electromagnetic charge around a magnetic monopole, as shown in figure 3.1. The minimal Lagrangian that couples a particle of charge Q to the electromagnetic gauge potential is:

$$\frac{1}{2}m\dot{x}_\mu(\tau)\dot{x}^\mu(\tau) - Q\dot{x}^\mu(\tau)A_\mu(\tau, x) \tag{3.6}$$

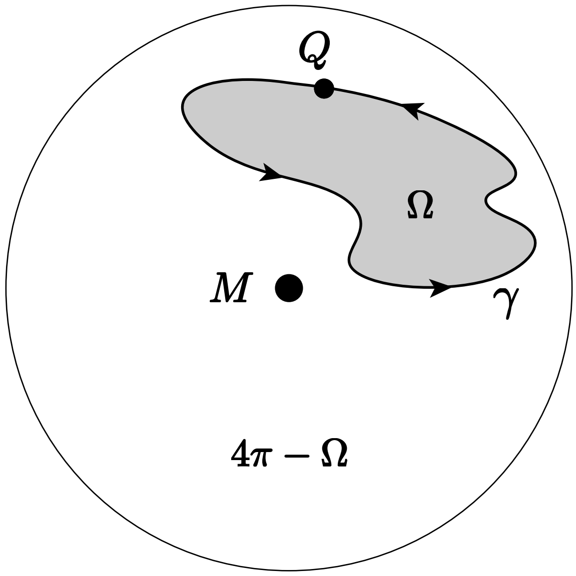

Figure 3.1: A point particle of electric charge Q moves in a path γ around a magnetic monopole of charge M. The path subtends a solid angle Ω.

Consider an infinitesimal path traced by an electrically charged particle. When the symmetry is gauged, the phase acquired by its wave function over a small path ds is [1]

$$(1 + iQA_\mu \frac{dx^\mu}{ds} ds) \tag{3.7}$$

which can be integrated along a loop γ to obtain

$$\exp\left(iQ \int_\gamma A_\mu \dot{x}^\mu \, ds\right) \tag{3.8}$$

The existence of a magnetic monopole of charge M implies that the magnetic flux through a sphere S enclosing it is

$$\int_S B \cdot dS = M \tag{3.9}$$

[1]This can be seen by considering the covariant derivative along the path.

Then equation (3.8) can be when evaluated by noticing $\gamma = \partial S'$ for some surface $S' \subset S$, which subtends the solid angle Ω:

$$\exp\left(iQ\oint_{\partial S'} A \cdot dx\right) = \exp\left(iQ\int_{S'} B \cdot dS\right) = \exp\left(iQM\frac{\Omega}{4\pi}\right) \tag{3.10}$$

But the surface S' shares its boundary with the surface S'' subtending solid angle of $4\pi - \Omega$, so:

$$iQM\frac{\Omega}{4\pi} = -iQM + iQM\frac{\Omega}{4\pi} \tag{3.11}$$

Where the orientation of the path must be flipped for the integral over S''. This is the Dirac quantisation condition:

Dirac quantisation

The product of electric and magnetic charge must be quantised in units of $2\pi\hbar$:

$$QM = 2\pi\hbar n \tag{3.12}$$

Where the factor of $\hbar$ has been restored. This condition can be interpreted as monopoles being the source of the quantisation of electric charge, or as a consistency condition between electric and magnetic charges.

3.2 Non-abelian gauge theories

The generalisation of electromagnetism to non-abelian gauge theories is a natural step. The standard model of particle physics is a non-abelian gauge theory with gauge group SU(3) × SU(2) × U(1) [33]., and the strong force in particular is easily described as an interaction between objects with colour charge. A non-abelian gauge theory is defined by a gauge group G and a connection A_μ in the Lie algebra $\mathfrak{g}$. The derivative is replaced by a covariant derivative $D_\mu = \partial_\mu - igA_\mu$, where g is the coupling constant of the theory.

3.2.1 Coupling charge to the gauge field

From the language developed in 2.2 we know that the "electric" colour charge Q lives in the weight lattice of the gauge group G, which means that it is an object in the dual space of the Cartan subalgebra $\mathfrak{h}^*$. Meanwhile we can gauge transform the gauge field to be an element of $\mathfrak{h}$ and write it as $A_\mu = A^i_\mu H^i$, where H^i are the generators of the Cartan subalgebra. Since a representation is a linear map and using equation (2.17) we can write:

$$\pi(A^i_\mu H^i)v = A^i_\mu q^i v \tag{3.13}$$

Thus the charge Q couples to the gauge potential as $\langle Q, A_\mu \rangle$, where the product is intended as the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$ defined in 2.3.

3.2.2 Gauge invariant charge

In 2.4 we explained how for non-abelian groups observable charges can be classified using representation theory. However, these observables are not invariant under the action of the group. When the symmetry is a gauge symmetry, these are not gauge invariant, while the gauge principle demands that only gauge invariant quantities are physical. In fact, colour charges are not even constant as the particle moves around space. The evolution of a particle's color charge in a non-Abelian gauge field is described by the Wong equations [34, 35], which govern the dynamics of a classical particle interacting with a gauge field. Specifically, the time evolution of the colour charge Q^a (where a labels the components of the colour charge in the adjoint representation of the gauge group) is given by:

$$\frac{dQ^a}{d\tau} = f^{abc} A^b_\mu Q^c \frac{dx^\mu}{d\tau}. \tag{3.14}$$

Here, f^{abc} are the structure constants of the non-Abelian gauge group, and A_μ^b are the components of the gauge field. This equation shows that the color charge Q^a precesses as the particle interacts with the gauge field, leading to a non-constant charge. The Wong equations make clear that color charge is not a fixed quantity for a particle moving through space, but rather, it evolves in a manner determined by the local gauge field configuration. Thus, there is a need for a gauge invariant way to probe charges in non-abelian gauge theories. This information can be encoded in the Wilson line operator, Wilson lines represent the holonomy of the gauge field along a path γ, they tell how the internal degrees of freedom of a particle evolve as it moves through space, linking spacetime motion to motion in the gauge group.

Wilson line

A Wilson line is a path ordered exponential of the gauge potential along a path γ with endpoints x and y:

$$W(x,y) = \mathcal{P}\exp\left(ig\int_x^y A_\mu dx^\mu\right) \tag{3.15}$$

Where g is the coupling constant of the theory. Wilson lines live in different representations of the gauge group G, as they are defined by the action of the gauge potential $A_\mu = A_\mu^a T^a$, where the T^a can be in any representation of the Lie algebra $\mathfrak{g}$. Wilson lines themselves form a group, the holonomy group, which is isomorphic to a subgroup of G [36, 37].

Wilson lines are in representations of the gauge group, and thus there is a correspondence between Wilson lines and charges in the theory. They can be used to construct a gauge invariant operator, the Wilson loop, which is the trace of a Wilson line along a closed path.

$$W(\gamma) = \mathrm{tr}\mathcal{P}\exp\left(ig\int_\gamma A_\mu dx^\mu\right) \tag{3.16}$$

The Wilson loop operator can be considered as the gauge phase acquired by an infinitely massive probe particle with colour charge around a loop γ. It is exactly the quantity obtained by integrating out the internal degrees of freedom of the particle [31].

′t Hooft lines

Similarly, there are objects that can be used to define magnetic charges, called ′t Hooft loops. As operators, they are harder to define, as they correspond to the restriction of the path integral along those loop paths for which the gauge field satisfies the condition:

$$\int_S B \cdot dS = M \tag{3.17}$$

Where B is the magnetic colour potential, M is the magnetic charge, and both are now elements of $\mathfrak{g}$. These can be interpreted as enforcing the existence of a monopole with charge M. A ′t Hooft loop operator is the magnetic analogue of the Wilson loop operator, and is also a gauge invariant object.

3.2.3 GNO quantisation

Using the tools developed so far, it is possible to derive a generalisation of the Dirac quantisation condition to non-abelian gauge theories. As was first done by Goddard, Nuyts and Olive in 1977 [38]. It can be easily inferred from that magnetic charges live in the Lie algebra $\mathfrak{g}$ like the gauge potential. We note that we can always gauge transform the gauge potential to the Cartan subalgebra $\mathfrak{h}$. This will similarly restrict the magnetic charge M to the Cartan subalgebra as well and we can write $M = m^i H^i$ We consider the presence of electric and magnetic charges as the insertion of Wilson and ′t Hooft lines in the path integral for a particle with electric charge Q. Starting from a Wilson line for charge Q we have:

$$\exp\left(iq^i \oint_{\partial S'} A^i \cdot dx\right) = \exp\left(iq^i \int_{S'} B \cdot dS\right) = \exp\left(iQ \cdot M \frac{\Omega}{4\pi}\right) \tag{3.18}$$

Giving the condition

$$Q \cdot M = 2\pi n \tag{3.19}$$

As seen in 2.2, the electric charge Q is an element of the weight lattice of the gauge group G, while we already know that the magnetic charge M is an element of the Lie algebra $\mathfrak{g}$. The condition above can be realised only for specific elements M of $\mathfrak{h}$, recalling property (2.29), we must have that the allowed magnetic charges are only multiples of the co-roots of the gauge group G. From the decomposition of representations of $\mathfrak{g}$ as $\mathfrak{h}$ representations we know that for and any weight μ in the weight lattice:

$$2\frac{\alpha \cdot \mu}{\alpha^2} \in \mathbb{Z} \tag{3.20}$$

defining the co-root $\alpha^\vee = \frac{\alpha}{\alpha^2}$ then we have $\mu \cdot \alpha \in \frac{1}{2}\mathbb{Z}$. This exactly solves the condition above by choosing $\mu_m = \alpha^\vee$. For a simple lie algebra $\mathfrak{g}$, the lattice spanned by the co-roots is also the root lattice of its dual $\mathfrak{g}^\vee$.

$$\Lambda_{\alpha^\vee}(\mathfrak{g}) = \Lambda_\alpha(\mathfrak{g}^\vee) \tag{3.21}$$

In the case of Lie groups in the A, D, E series $\mathfrak{g} = \mathfrak{g}^\vee$. Hence, the magnetic charges must sit in the co-root lattice or $\mathfrak{g}$ or in the root lattice of $\mathfrak{g}^\vee$. This can be understood as a condition on the allowed $'$t Hooft line operators compatible with Wilson line operators.

SU(N) vs SU(N)/Z$_N$ again

As dicussed in Subsection in 2.4.1, there are more than one theories with the same Lie algebra, which differ in the global structure by the quotient of the centre of the group. This matters for the quantisation above, as for the group $\mathrm{SU}(N)/\mathrm{Z}_N$ the allowed representations are only tensor products of the adjoint representation, which is invariant under the centre Z_N. Thus we could think of the electric charge as a vector in the Cartan subalgebra $\mathfrak{h}^\vee$ of $\mathfrak{g}^\vee$, that is, as a co-root of the dual algebra $\mathfrak{g}^\vee$. This lessens the requirement on the magnetic charge, which now lives in the root lattice of $\mathfrak{g}^\vee$. In this theory, magnetic and electric charges swap roles. Magnetic charges now live in the weight lattice $\Lambda_\mu(\mathfrak{g}^\vee)$, while electric charges live in the root lattice $\Lambda_\alpha(\mathfrak{g})$.

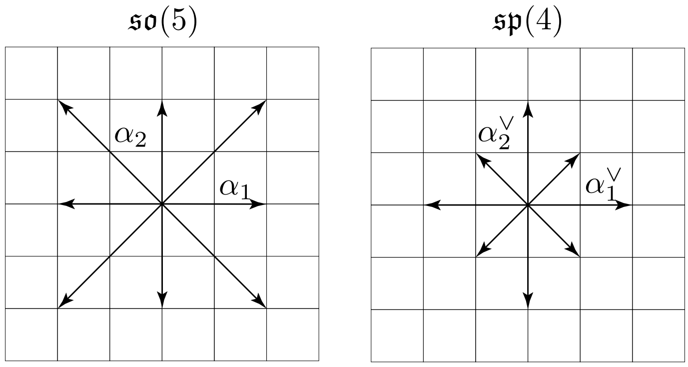

Figure 3.2: The Lie algebras of groups in the B_n, and C_n series are dual to each other: the co-roots of B_n are the roots of C_n and vice versa. In the image, we have the root systems of B_2 and C_2, also known as $\mathfrak{so}(5)$ and $\mathfrak{sp}(4)$ respectively. These are accidentally isomorphic $\mathfrak{so}(5) \cong \mathfrak{sp}(4)$, but the same is not generally true for the other Lie groups in the two series.

Chapter 4

Final remarks

4.1 Summary

In this work, we have investigated how charge is understood thanks to representation theory. For any symmetry of a theory, we have seen that there is a corresponding charge, a quantity that is conserved. Classes of particles in the theory can be classified according to the representations of the symmetry group, while their specific charges are determined by their decomposition in modules of the cartan subalgebra. Charged states can be understood as elements of weight spaces, labelled by weights, which are elements of the dual space of the Cartan subalgebra. These naturally provide a set of compatible time-independent observable. This led naturally to the quantisation of charge in the case of compact symmetry groups. In gauge theories, we have discussed the difference between electric and magnetic charges, and the properties of magnetic monopoles, which are at first glance excluded by the Bianchi identities. In the simple electromagnetic case, these can be realised with singular gauge potentials, like a Dirac string [2, 28], or through topological solitons ì [4, 3]. However, a statement about the compatibility of electric and magnetic charges can be made regardless of how the monopole is realised, leading to the Dirac quantisation condition. Then, we regarded the analogue of magnetic monopoles in non-abelian gauge theories. We were able to make statements about the representation theory of these monopoles themselves, establishing that on pure representation-theoretic grounds, these monopoles can exist when the colour-electric charges obey generalised quantisation conditions. These conditions can either be interpreted a la Dirac to infer the quantisation of charge from the existence of a monopole, or as a statement about the compatibility of electric and magnetic charges. Furthermore, they show how magnetic charges are intimately related to the nonlocal structure of the gauge theory, allowing to differentiate between gauge groups with different topological structures.

4.2 Last words on gravitation

The case of gravity is of particular interest for future investigation. The structure of gravity as a gauge theory is not well understood, with two different approaches leading to quite different results. By considering the metric as a spin-2 gauge field, there has been some success in the linearised regime, leading to similar predictions as the electromagnetic case [39]. On the other hand we have Einstein-Cartan theory, which can be understood as a gauge theory through the symmetry breaking of an $\mathrm{SO}(4,1)$ gauge field through a mechanism explained by MacDowell and Mansouri [40, 41]. This approach seems more promising to understand the non-linear regime of gravity: by considering massive particles as gravitational Wilson lines [42], the methods discussed in this work could be applied to the study of gravitational magnetic charges and mass quantisation. We conclude presenting a possible reason why the investigation of gravitational magnetic monopoles is of interest: as seen in 3, monopoles can emerge in non-topologically trivial configurations. It is also known that monopoles are present in lattice theories [43]. Since there are reasons to believe that spacetime is in some way fundamentally discrete, the existence of such monopoles could be a consequence of quantum gravity, and detection could offer important clues into the structure of spacetime itself.

Bibliography

[1] P. Curie, “Sur la possibilité d’existence de la conductibilité magnétique et du magnétisme libre,” *Journal de Physique Théorique et Appliquée*, vol. 3, no. 1, pp. 415–417, 1894.

[2] P. A. M. Dirac, “Quantised singularities in the electromagnetic field,” *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 133, no. 821, pp. 60–72, 1931. Publisher: The Royal Society.

[3] A. M. Polyakov, “Particle spectrum in quantum field theory,” *JETP Lett.*, vol. 20, pp. 194–195, 1974.

[4] G. t. Hooft, “Magnetic monopoles in unified gauge theories,” *Nuclear Physics*, 1974.

[5] P. B. Price, E. K. Shirk, W. Z. Osborne, and L. S. Pinsky, “Evidence for detection of a moving magnetic monopole,” *Phys. Rev. Lett.*, vol. 35, pp. 487–490, Aug 1975.

[6] B. Cabrera, “First results from a superconductive detector for moving magnetic monopoles,” *Phys. Rev. Lett.*, vol. 48, pp. 1378–1381, May 1982.

[7] K. A. Milton, “Theoretical and experimental status of magnetic monopoles,” *Reports on Progress in Physics*, vol. 69, p. 1637–1711, May 2006.

[8] A. Rajantie, "Magnetic monopoles in field theory and cosmology," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 370, pp. 5705–5717, Dec. 2012. Publisher: Royal Society.

[9] N. S. Craigie, P. Goddard, and W. Nahm, "Monopoles in quantum field theory: proceedings of the monopole meeting, trieste, italy december 1981," *(No Title)*, 1982.

[10] R. W. Jackiw, "Dirac's magnetic monopoles (again)," in *Proceedings of the Dirac Centennial Symposium*, p. 137–143, WORLD SCIENTIFIC, Dec. 2003.

[11] R. Foot, H. Lew, and R. R. Volkas, "Electric Charge Quantization," *Journal of Physics G: Nuclear and Particle Physics*, vol. 19, pp. 361–372, Mar. 1993. arXiv:hep-ph/9209259.

[12] C. N. Yang, "Charge quantization, compactness of the gauge group, and flux quantization," *Phys. Rev. D*, vol. 1, p. 2360, 1970.

[13] J. Preskill, "Magnetic monopoles," *Annual Review of Nuclear and Particle Science*, 1984.

[14] J. P. Preskill, "Cosmological production of superheavy magnetic monopoles," *Phys. Rev. Lett.*, vol. 43, pp. 1365–1368, Nov 1979.

[15] T. W. B. Kibble, "Topology of cosmic domains and strings," *Journal of Physics A: Mathematical and General*, vol. 9, p. 1387, aug 1976.

[16] A. H. Guth, "Inflationary universe: A possible solution to the horizon and flatness problems," *Phys. Rev. D*, vol. 23, pp. 347–356, Jan 1981.

[17] A. Albrecht and P. J. Steinhardt, "Cosmology for grand unified theories with radiatively induced symmetry breaking," *Phys. Rev. Lett.*, vol. 48, pp. 1220–1223, Apr 1982.

[18] L. B. Okun and W. J. Marciano, *Leptons and Quarks*. 1984.

[19] H. Weyl, *The Theory Of Groups And Quantum Mechanics*. 1931.

[20] P. Woit, *Quantum Theory, Groups and Representations*. 2017.

[21] W.-K. Tung, *Group Theory In Physics: An Introduction To Symmetry Principles, Group Representations, And Special Functions In Classical And Quantum Physics*. 2020.

[22] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. 2004.

[23] T. Hawkins, *The Doctoral Thesis of Élie Cartan*, pp. 182–224. New York, NY: Springer New York, 2000.

[24] D. J. Griffiths, *Introduction to electrodynamics*. Pearson, 2013.

[25] M. Gell-Mann, "The interpretation of the new particles as displaced charge multiplets," *Il Nuovo Cimento (1955-1965)*, vol. 4, pp. 848–866, Apr. 1956.

[26] J. P. Gauntlett, J. A. Harvey, and J. T. Liu, "Magnetic monopoles in string theory," *Nuclear Physics B*, vol. 409, p. 363–381, Nov. 1993.

[27] D. J. Gross and M. J. Perry, "Magnetic monopoles in kaluza-klein theories," *Nuclear Physics B*, vol. 226, no. 1, pp. 29–48, 1983.

[28] P. Dirac, "The theory of magnetic poles," 1948.

[29] E. Corrigan and D. Olive, "Colour and magnetic monopoles," *Nuclear Physics B*, vol. 110, no. 2, pp. 237–247, 1976.

[30] R. Heras, "Dirac quantisation condition: a comprehensive review," *Contemporary Physics*, vol. 59, p. 331–355, Oct. 2018.

[31] D. Tong, "Gauge theory," 2018. Lecture Notes, Department of Applied Mathematics and Theoretical Physics, University of Cambridge.

[32] Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in the quantum theory," *Phys. Rev.*, vol. 115, pp. 485–491, Aug 1959.

[33] S. Weinberg, *The Quantum Theory of Fields, Volume 1: Foundations.* Cambridge University Press, 2005.

[34] S. K. Wong, "Field and particle equations for the classical Yang-Mills field and particles with isotopic spin," *Il Nuovo Cimento A (1965-1970)*, vol. 65, pp. 689–694, Feb. 1970.

[35] S. N. Storchak, "Wong's equations in yang-mills theory," *Central European Journal of Physics*, 2014.

[36] M. Nakahara, *Geometry, Topology and Physics, Second Edition.* Graduate student series in physics, 2nd ed ed., 2003.

[37] M. J. Hamilton, "Mathematical gauge theory," *Universitext*, 2017.

[38] P. Goddard, J. Nuyts, and D. I. Olive, "Gauge theories and magnetic charge," *Nuclear Physics*, 1977.

[39] A. Zee, "Gravitomagnetic pole and mass quantization," *Phys. Rev. Lett.*, vol. 55, pp. 2379–2381, Nov 1985.

[40] S. W. MacDowell and F. Mansouri, "Unified geometric theory of gravity and supergravity," *Phys. Rev. Lett.*, vol. 38, pp. 739–742, Apr 1977.

[41] D. K. Wise, "Macdowell-mansouri gravity and cartan geometry," *Classical and Quantum Gravity*, 2010.

[42] L. Freidel, J. Kowalski-Glikman, and A. Starodubtsev, "Particles as wilson lines of the gravitational field," *Physical Review D*, 2006.

[43] B. L. G. Bakker, A. I. Veselov, and M. A. Zubkov, "Monopoles in lattice electroweak theory," 2008.

[44] J. Schwichtenberg, *Physics from Symmetry.* Undergraduate Lecture Notes in Physics, Springer, 2nd ed., 2018.

[45] J. Schwinger, "Sources and magnetic charge," 1968.

[46] L. Kampmeijer, "On a unified description of non-abelian charges, monopoles and dyons," 2009.

[47] M. Blagojevic and P. Senjanović, "The quantum field theory of electric and magnetic charge," *Physics Reports*, 1988.

[48] J. Leach, "An elementary introduction to groups and representations," 2000.

[49] W. J. Marciano and H. Pagels, "Classical su(3) gauge theory and magnetic monopoles," *Physical Review D*, 1975.

[50] O. Aharony, N. Seiberg, and Y. Tachikawa, "Reading between the lines of four-dimensional gauge theories," *Journal of High Energy Physics*, vol. 2013, Aug. 2013.

[51] P. Korcyl, M. Koren, and J. Wosiek, "Wilson loops with arbitrary charges," *Acta Phys. Polon. B*, vol. 46, no. 2, p. 247, 2015.

[52] J. Schwichtenberg, "Physics from symmetry," 2015.

[53] B. C. Hall, "Lie groups, lie algebras, and representations: An elementary introduction," 2004.

[54] R. Jackiw, "Topological aspects of gauge theories," *arXiv: High Energy Physics - Theory*, 2005.

[55] P. Gomes, "An introduction to higher-form symmetries," *SciPost Physics Lecture Notes*, 2023.

[56] P. Goddard and D. I. Olive, "Magnetic monopoles in gauge field theories," *Reports on Progress in Physics*, 1978.

[57] Y. Shnir, "Magnetic monopoles," *Physics of Particles and Nuclei Letters*, 2011.

[58] G. t. Hooft, "A property of electric and magnetic flux in non-abelian gauge theories," *Nuclear Physics*, 1979.

[59] A. P. Balachandran, S. Borchardt, and A. Stern, "Lagrangian and hamiltonian descriptions of yang-mills particles," *Physical Review D*, 1978.

[60] N. Christ, "Theory of magnetic monopoles with non-abelian gauge symmetry," 1975.

[61] B. C. Hall, "An elementary introduction to groups and representations," *arXiv: Mathematical Physics*, 2000.

[62] F. Bais and B. J. Schroers, "Quantisation of monopoles with non-abelian magnetic charge," *Nuclear Physics*, 1998.

[63] M. Saha, "Note on dirac's theory of magnetic poles," 1949.

[64] P. Goddard and D. I. Olive, "Charge quantization in theories with an adjoint representation higgs mechanism," *Nuclear Physics*, 1981.

[65] M. Nosrati, "On the magnetic current density in maxwell's equations based on noether's theorem," 2000.

[66] T. Wu and C. Yang, "Concept of nonintegrable phase factors and global formulation of gauge fields," 1975.

[67] J. Paczos, K. Debski, S. Cedrowski, S. Charzy'nski, K. Turzy'nski, A. Ekert, and A. Dragan, "Covariant quantum field theory of tachyons," *Physical Review D*, 2023.

[68] L. Kampmeijer, J. Slingerland, B. J. Schroers, and F. Bais, "Magnetic charge lattices, moduli spaces and fusion rules," *Nuclear Physics*, 2009.